Sample Path Analysis Methods

Outline :

- Motivation (Appetizer)
  - Performance Bounds
  - Proving Optimality of Scheduling Policies
- Technical Development (Fish Course)
  - Strong stochastic Ordering
    - Application to $GI/GI/1$ Queue.
  - Convex Ordering
    - Application to $GI/GI/1$ Queue.
  - Majorization and stochastic Ordering

    * Application to Load Balancing.

- Applications (Meat Course)
  - Optimality of SRPT policy
  - Fair scheduling policies
  - Optimality of earliest deadline scheduling
- Advantages and Disadvantages (Dessert)
- Summary (Coffee)

What is sample path analysis ?

- Illustration

- Study system behavior given arrival , service times as function of
  - Buffer size (B)
  - Number of servers (C)

  - Buffer management policy

- <u>What can we get out of SPA ?</u>
  - Aid in performance evaluation
  - Prove Optimality of scheduling policies

Performance Evaluations

- Stability Condition
- Performance Bounds
- Performance properties : Symmetry of throughput for tandem queues with finite buffers

- Idea : guess optimal policy , compare behavior w/ arbitrary policy on a sample path

- Compare What? need a suitable system state
- How to Compare?
  1. Strict inequality
  2. Majorization
  3. set dominance
- How to Couple ? Need an i.i.d. service times,exponential service time

Stochastic Ordering

- X, Y be real valued random variables
- Y is stochastically larger than $X (X \leq_{st} Y)$ iff:

$$\begin{aligned} Prob[Y < t] &\leq Prob[X < t] \quad \text{for all t} \quad \text{or} \\ Prob[Y \geq t] &\geq Prob[X \geq t] \quad \text{for all t} \quad \text{or} \\ E[f(X)] &\leq E[f(Y)] \quad \text{for all increasing f} \end{aligned}$$

- Strassen's Theorem

- If $X \leq_{st} Y$,then there exist r.v's $\tilde{X}, \tilde{Y}$ such that $\tilde{X} =_{st} X, \tilde{Y} =_{st} Y$ and almost surely,

$$\tilde{X} \leq \tilde{Y}$$

Properties of stochastic Ordering

($X \leq_{st} Y$ iff $Prob[Y < t] \leq Prob[X < t], E[f(X)] \leq E[f(Y)]$ for all increasing f

- Property (1) If $X \leq_{st} Y$,then $E[X] \leq E[Y]$
- Property (2) If $X \leq_{st} Y$,then $-Y \leq_{st} -X$
- Property (3) Let $X_1, X_2, \cdots X_n$and $Y_1, Y_2, \cdots, Y_n$ be two sequences of independent r.v's . If $X_i \leq_{st} Y_i$ for all i ,then

$$f(X_1, X_2, \cdots, X_n) \leq f(Y_1, Y_2, \cdots, Y_n)$$

provided f is an increasing function.

- Example:

$$
\begin{aligned}
f(X_1, X_2, \cdots, X_n) &= \sum_{i=1}^{n} X_i \\
f(X_1, X_2, \cdots, X_n) &= MAX_i(X_i) \\
f(X_1, X_2, \cdots, X_n) &= [MAX_i(X_i)]^+
\end{aligned}
$$

Application to GI/GI/1 Queue

- Description of GI/GI/1 queue
  - $\{X_i\}_{i=0}^{\infty}$ sequence of i.i.d interarrival times,
  - $\{S_i\}_{i=0}^{\infty}$ sequence of i.i.d service times,
  - W_i = wait time of customer $i, i = 1, 2, \cdots$

$$
\begin{aligned}
W_{i+1} &= (W_i + S_i - X_{i+1})^+ \quad i \geq \phi \\
W_0 &= 0 \\
S_0 &= 0
\end{aligned}
$$

- Result : Consider two GI/GI/1 queue,the j^{th} queue has interarrival times $\{X_i^j\}$,service times $\{S_i^j\}$ and wait time $\{W_i^j\}, j = 1, 2$.If $X_i^1 \leq_{st} X_i^2$ and $S_i^2 \leq_{st} S_i^1$

$$W_i^2 \leq_{st} W_i^1$$

- Proof:
  - Given : $X_i^1 \leq_{st} X_i^2, S_i^2 \leq_{st} S_i^1$
  - By induction on i
  - Trivially tree for $i = 0$
  - Assume $W_i^2 \leq_{st} W_i^1$ for $i \leq n$,we have

$$\begin{aligned}
-X_{n+1}^2 &\leq_{st} -X_{n+1}^1 \qquad \text{(property 2)} \\
W_n^2 + S_n^2 - X_{n+1}^2 &\leq_{st} W_n^1 + S_n^1 - X_{n+1}^1 \qquad \text{(property 3)} \\
(W_n^2 + S_n^2 - X_{n+1}^2)^+ &\leq_{st} (W_n^1 + S_n^1 - X_{n+1}^1)^+ \qquad \text{(property 3)} \\
W_{n+1}^2 &\leq_{st} W_{n+1}^1
\end{aligned}$$

- all i^{th}moments,$i \geq i$ of system 2 are smaller than system 1.

Properties of convex order ($\leq_{cx}$) and Increasing convex order ($\leq_{icx}$)

- Property 1 : If $X \leq_{icx} Y \text{and} E[X] = E[Y], \text{then} X \leq_{cx} Y$
- Property 2 : If X and Y are non-negative r.v's and $X \leq_{icx} Y$ then $E[X^i] \leq E[Y^i] \quad , i = 1, 2, \cdots$
- Property 3 : If $X \leq_{cx} Y \rightarrow E[X] = E[Y], \sigma_X \leq \sigma_Y, E[X^{2i}] \leq E[Y^{2i}], i = 1, 2, 3, \cdots$
- Property 4 : If $X \leq_{cx} Y \rightarrow -X \leq_{cx} -Y$
- Property 5 : If $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ be two sequences of independent r.v's.If $X_i \leq_{icx} Y_i$ for all i,then

$$f(X_1, X_2, \cdots, X_n) \leq_{icx} f(Y_1, Y_2, \cdots, Y_n)$$

provided f is an increasing convex function

- Example : $f(.) = \sum_{i=1}^{n} X_i \rightarrow$ convex function
  $f(.) = max_i X_i \rightarrow$ increasing convex function
  $f(.) = (max_i X_i)^+ \rightarrow$ increasing convex function

Application to GI/GI/1 Queue

- Given : arrival times $\{X_i^j\}$,service times $\{S_i^j\}$ and wait times $\{W_i^j\}$
  If $X_i^1 \leq_{cx} X_i^2; S_i^1 \leq_{icx} S_i^2; E[X_i^1] = E[X_i^2]$
  then $W_i^1 \leq_{icx} W_i^2$
  note $W_{i+1} = (W_i + S_i - X_{i+1})^+; W_0 = 0$

- Proof :
  - By induction on i
  - Trivial on i

– Assume $W_i^1 \leq_{icx} W_i^2$ for $i \leq n$,we have

$$\begin{aligned}
-X_{n+1}^1 &\leq_{cx} -X_{n+1}^2 \\
-X_{n+1}^1 &\leq_{cx} -X_{n+1}^2 \\
(W_n^1 + S_n^1 - X_{n+1}^1)^+ &\leq_{cx} (W_n^2 + S_n^2 - X_{n+1}^2)^+ \\
W_{n+1}^1 &\leq_{cx} W_{n+1}^2
\end{aligned}$$

Majorization

- $X = (X_1, X_2, \cdots, X_n), Y = (Y_1, Y_2, \cdots, Y_n) \in \Re^n$
- $\hat{X}_i, \hat{Y}_i$ denote the i^{th} largest elements of X, Y
- Ymajorizes$X(X < Y)$ iff

$$\begin{aligned} \sum_{i=1}^{k} \hat{X}_i &\leq \sum_{i=1}^{k} \hat{Y}_i, k = 1, 2, \cdots, \underline{n-1} \\ \sum_{i=1}^{n} \hat{X}_i &= \sum_{i=1}^{n} \hat{Y}_i \end{aligned}$$

- Y weakly majorizes $X(X <_w Y)$iff

$$\sum_{i=1}^{k} \hat{X}_i \leq \sum_{i=1}^{k} \hat{Y}_i, k = 1, 2, \cdots, \underline{n}$$

What preserves majorization?

Operations

- If $X = (X_1, X_2, \cdots, X_n)$ and $Y = (Y_1, Y_2, \cdots, Y_n)$,then

$$X \circ Y = (X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n)$$

- If $X = (X_1, X_2, \cdots, X_n)$ where $= \hat{X}_k = X_i$,then

$$X + c\hat{e_k} = (X_1, X_2, \cdots, X_i + c, \cdots, X_n)$$

- If $X_1, Y_1 \in \Re^n, X_2, Y_2 \in \Re^m$ and $X_i < Y_i i = 1, 2, \cdots$ then

$$X_1 \circ X_2 < Y_1 \circ Y_2$$

- If $X, Y \in N^n$, $X < Y (X <_w Y)$,then

$$\begin{array}{rccl} X + \hat{e}_l & < (<_w) & Y + \hat{e_k} & 1 \leq k \leq l \leq n \\ X - \hat{e}_l & < (<_w) & Y - \hat{e_k} & 1 \leq k \leq l \leq n \\ X - \hat{e}_l & <_w & Y & 1 \leq l \leq n \\ X & <_w & Y - \hat{e_k} & 1 \leq k \leq n \end{array}$$

provided $\sum_{i=1}^{j} \hat{X}_i < \sum_{i=1}^{j} \hat{Y}_i, j = k, \cdots, n$

Schur convex Function

- $\phi : \Re^n \to \Re$ is a Schur convex function if

$$\phi(X) \leq \phi(Y) \text{ for all } X, Y \in \Re^n \text{ such that } X < Y$$

- $\phi : \Re^n \to \Re$ is a weak Schur convex function if

$$\phi(X) \leq \phi(Y) \text{ for all } X, Y \in \Re^n \text{ such that } X <_w Y$$

- ϕ is a weak Schur convex function iff ϕ is an increasing Schur convex function
- Example:

$$\phi(X) = max_i X_i$$

$$
\begin{aligned}
\phi(.) &= \sum_{i=1}^{n} f_i(X_i) \quad \forall \text{convex function} f_i \\
\phi(.) &= \text{no. of zeros if } X_i \in \Re^{+} \\
\phi(.) &= \sum_{i=1}^{n} 1\{X_i = 0\} \quad \text{if } X_i \geq 0
\end{aligned}
$$

Schur Convex Ordering

- $X, Y \in \Re^n$(or $\in N^n, (\Re^+)^n$) are random variables.
- Y is larger than X in the sense of Schur convex order ($X \leq_{scx} Y$) iff

$$E[\phi(X)] \leq E[\phi(Y)] \quad \forall \text{ Schur convex} \phi$$

- Y is larger than X in the sense of increasing Schur convex order ($X \leq_{scx} Y$) iff

$$E[\phi(X)] \leq E[\phi(Y)] \quad \forall \text{ increasing Schur convex} \phi$$

Properties of $\leq_{scx}$ and $\leq_{iscx}$

- Property 1 : $X \leq_{scx} Y$ iff $\phi(X) \leq_{st} \phi(Y), \forall$ Schur convexϕ
- Property 2 :
  $X \leq_{scx} Y$ iff $\phi(X) \leq_{st} \phi(Y), \forall$ increasing Schur convexϕ
- Property 3(Coupling theorem) : If $X \leq_{scx} Y (X \leq_{iscx} Y)$,then there exist random variable $\tilde{X}$ and $\tilde{Y}$ such that

$$\tilde{X} =_{st} X, \tilde{Y} = {}_s tY, \tilde{X} < \tilde{Y}$$

Application 1 : Optimality of SRPT policy

- General arrivals $a_1 < a_2 < \cdots$ Number of arrivals by time $t > 0$ is $\underline{A(t)}$
- General services time $\{S_i\}$
- $\sum$ - class of non-idling , preemptive policies where service time information is available
- SRPT - policy that serves job with smallest remaining processing time .
  <u>state variables , metrics</u>

- $N_\pi(t)$ =number of jobs in system at time t under policy π
- N_π = stationary value of $N_\pi(t)$, if it exists.
- R_π = stationary response time under π
- $X_\pi(t) = (x_\pi\prime(t), \cdots, x_\pi^{A(t)}(t))$where$X_\pi^i(t)$ is the remaining processing time of the i^{th} job.

Optimality of SRPT

- SRPT is optimal ,i.e.

$$N_{SRPT}(t) \leq_{st} N_{\pi}(t), t \geq 0, \forall \pi \in \sum$$

  implies $R_{SRPT} \leq_{st} R_{\pi} \forall \pi \in \sum$ if N_{SRPT} and N_{π} exist.

- Proof:Given a fixed arrival sequence $a_1, a_2, \cdots$

$$X_{\pi}(t) \leq_{scx} X_{SPRT}(t) \qquad t \geq 0, \forall \pi \in \sum$$

  Optimality of SRPT follows than
  $\rightarrow$ fix arrival times and apply lemma above
  $\rightarrow Z(X_{\pi}(t) \leq_{st} Z(X_{SRPT}(t))$
  $\rightarrow A(t) - Z(X_{SRPT}(t) \leq_{st} A(t) - Z(X_{\pi}(t)) \rightarrow$ Remove conditioning on arrival times

Detail Proof:

- Pick your favorite policy π
- Fix service times will show that $X_\pi(t) < X_{SRPT}(t), t > 0$
- By induction on times of events , $t_1 < t_2 < \cdots$;events - arrival , completion or preemption under π
- Basis step : trivial $X(t_1) = X_{SRPT}(t_1) = (S_1)$
- Induction step : assume true for $t \le t_i$ extend to t_{i+1}
  - For $t_i \le t < t_{i+1}$ the basic equation is

$$\begin{aligned} X_\pi(t) &= X_\pi(t_i) - (t - t_i)\hat{e_k} \\ X_{SRPT}(t) &= X_{SRPT}(t_i) - (t - t_i)\hat{e}_{L_{SRPT}} \end{aligned}$$

for some $k \le L_{SRPT}(t)$

- claim $X_\pi < X_{SRPT}(t)$
  - If event is preemption , service completion , set $t = t_{i+1}$
  - arrival of job n

$$\begin{aligned} X_\pi(t_{i+1}) &= X_\pi(t_{i+1}^-) \circ (\sigma_n) \\ X_{SRPT}(t_{i+1}) &= X_{SRPT}(t_{i+1}^-) \circ (\sigma_n) \end{aligned}$$

$\rightarrow$ preserves majorization

- Therefore,

$$X_\pi(t) < X_{SRPT}(t) \quad \forall t > \phi$$

Remove the arrival and service time condition

- Let $Z(.)$ be a function that $Z(.) = \sum_{i=1}^{\infty} 1[X_i = 0]$
Z is a Schur convex function $\rightarrow$

$$\begin{aligned}
Z(X_\pi(t)) &\leq_{scx} Z(X_{SRPT}(t)) \\
Z(X_\pi(t)) &\leq_{st} Z(X_{SRPT}(t)) \\
-Z(X_{SPRT}(t)) &\leq_{st} -Z(X_\pi(t)) \\
A(t) - Z(X_{SPRT}(t)) &\leq_{st} A(t) - Z(X_\pi(t)) \\
N_{SRPT}(t) &\leq_{st} N_\pi(t)
\end{aligned}$$