Transient Analysis

- Motivation of why we need to perform *transient analysis.*
- Important questions are:
  - state of the model at the end of a time interval,
  - the time until an event occurs,
  - the residence time in a subset of states during a given interval,
  - the number of given events in an interval.

General Solution

Let $\{X(t), t \geq 0\}$ be a CTMC with finite state space $\mathcal{S} = \{s_i : i = 1, \ldots, M\}$ and $\boldsymbol{Q}$ the transition rate matrix:

$$\boldsymbol{Q} = \begin{bmatrix} -q_1 & q_{12} & \cdots & q_{1M} \\ q_{21} & -q_2 & \cdots & q_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ q_{M1} & q_{M2} & \cdots & -q_M \end{bmatrix} \tag{1}$$

where $q_i = \sum_{j=1, j \neq i}^{M} q_{ij}$.

Let $\boldsymbol{\Pi}(t)$ be a $M \times M$ matrix where $\pi_{ij}(t)$ in $\boldsymbol{\Pi}(t)$ is :

$$\pi_{ij}(t) = P[X(t) = s_j | X(0) = s_i] \tag{2}$$

Based on Kolmogorov's forward equation, we have:

$$\mathbf{\Pi}'(t) = \mathbf{\Pi}(t)\boldsymbol{Q} \tag{3}$$

Solving the above matrix equation, we have:

$$\mathbf{\Pi}(t) = \boldsymbol{e}^{\boldsymbol{Q}t} \tag{4}$$

Let $\boldsymbol{\pi}(t) = [\pi_1(t), \ldots, \pi_M(t)]$ be a $1 \times M$ row vector such that $pi_i(t)$ equal to the $P[X(t) = s_i]$. Therefore, we have:

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\mathbf{\Pi}(t) \tag{5}$$

In general, finding $\boldsymbol{\pi}(t)$ involves finding the corresponding eigenvalues and eigenvectors of $\boldsymbol{Q}$, which is computationally difficult.

Uniformization Method

Uniformization is a computational efficient method of performing transient analysis. Given a homogeneous CTMC $X(t)$ and the corresponding rate matrix $\boldsymbol{Q}$, let us define a homogeneous DTMC $X'(n)$ with the one-step transition probability matrix $\boldsymbol{P}$ as:

$$\boldsymbol{P} \quad = \quad \boldsymbol{I} + \frac{\boldsymbol{Q}}{\Lambda} \tag{6}$$

where $\Lambda \geq \max_i\{q_i\}$, i.e., Λ is greater than or equal to the absolute diagonal value in $\boldsymbol{Q}$. Therefore, we have:

$$\boldsymbol{\Pi}(t) = \boldsymbol{e}^{\boldsymbol{Q}t} = \boldsymbol{e}^{(\boldsymbol{P}-\boldsymbol{I})\Lambda\, t} = \boldsymbol{e}^{\boldsymbol{P}\Lambda t} e^{-\Lambda t} = \sum_{n=0}^{\infty} \boldsymbol{P}^n \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} \tag{7}$$

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\boldsymbol{\Pi}(t) = \sum_{n=0}^{\infty} \boldsymbol{\pi}(0)\, \boldsymbol{P}^n \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} = \sum_{n=0}^{\infty} \boldsymbol{\pi}(n) \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} \tag{8}$$

Probabilistic Interpretation

Consider a homogeneous CTMC X with state space $\mathcal{S}$ and M states and rate matrix $\boldsymbol{Q}$. The diagonal element q_i in $\boldsymbol{Q}$ represents the *output* rate (which is exponential). That is, once in s_i, the process moves to a different state with an exponential rate q_i. Furthermore, when it moves to other state, it moves to state s_j with probability q_j/q_i.

We construct a DTMC X' with one step transition probability matrix $\boldsymbol{P}$ (where $\boldsymbol{P} = \boldsymbol{I} + \boldsymbol{Q}/\Lambda$). We construct X' from X such that:

- X' has the same state space as X.
- The residence time in *any* state before a transition occurs is exponential with rate Λ.

- The probability that X' make a transition from s_i to s_j $(i \neq j)$ is equal to q_{ij}/Λ. Furthermore, X' make a transition back to the same state with probability $(1 - q_i/\Lambda)$.

Note that the probability that X' moves from s_i to s_j (for $i \neq j$) given that the transition is for a different state from s_i is:

$$\frac{q_{ij}/\Lambda}{q_i/\Lambda} = \frac{q_{ij}}{q_i}$$

which is equal to the probability that X goes from s_i to s_j in a transition.

Given that the process makes n transitions to the same state s_i before leaving it, the distribution of the residence time in this state is simply the sum of n random variables with exponential distribution and rate Λ. This is equal to the Erlangian-n distribution and the density function $E'_{n,\Lambda}(t)$ is given by:

$$E'_{n,\Lambda}(t) = \frac{\Lambda\,(\Lambda t)^{n-1}\,e^{-\Lambda t}}{(n-1)!}$$

The probability that the process makes $n-1$ self transitions before leaving s_i is:

$$\left(1-\frac{q_i}{\Lambda}\right)^{n-1}\frac{q_i}{\Lambda}$$

Combining the last two expression (via theorem of total probability), then we see that the *total residence time* in s_i has an exponential distribution with rate q_i.

Since the total residence time and transition probabilities of X and X' are the same, we can make the following conclusion:

$$\pi'_{ij}(t) = P[X'(t) = s_j | X'(0) = s_i] = P[X(t) = s_j | X(0) = s_i] = \pi_{ij}(t)$$

Therefore, the process CTMC X is *equivalent* to a DTMC X' subordinated to a Poisson process.

Practical Issues

Therefore, given $\boldsymbol{Q}$, we find $\boldsymbol{\Pi}(t)$, solve for $\boldsymbol{\pi}(n) = \boldsymbol{\pi}(0)\boldsymbol{P}^n$ and then *weighted* it by the probability that in the interval t, there are n Poisson arrival events.

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\boldsymbol{\Pi}(t) = \sum_{n=0}^{\infty} \boldsymbol{\pi}(0)\, \boldsymbol{P}^n \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} = \sum_{n=0}^{\infty} \boldsymbol{\pi}(n) \frac{(\Lambda t)^n}{n!} e^{-\Lambda t} \quad (9)$$

So, given $\boldsymbol{P}$, find $\boldsymbol{\pi}(n)$, which is just a vector matrix multiplication. Since in most cases, $\boldsymbol{P}$ is a sparse matrix, therefore, it is efficient to compute. Then weighted the X' via an Poisson probability that there will be n Poisson arrival events.

To avoid the infinite summing series, we can use *truncation* and at the same time, know the error in advance! For example, if we only consider $N+1$ transitions, we have:

$$\boldsymbol{\pi}(t) = \sum_{n=0}^{N} \pi(n) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} + \epsilon(N) \tag{10}$$

where $\epsilon(N)$ is the error when the series is truncated after N terms. An important advantage of the uniformization method is that it is possible to find N in advance for a given tolerance since $||\boldsymbol{\pi}(n)||_\infty \leq 1$ or:

$$\begin{aligned}
\epsilon(N) &= \sum_{n=N+1}^{\infty} \boldsymbol{\pi}(n) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \\
&\leq \sum_{n=N+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} = 1 - \sum_{n=0}^{N} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}
\end{aligned} \tag{11}$$

Extension to Non-homogeneous CTMC

Consider a non-homogeneous CTMC that has *different* transition rates for different interval of times. That is, for each interval $[t_{i-1}, t_i)$, the corresponding rate matrix is $\boldsymbol{Q}_i$ (we are assuming that X_i to be time homogeneous). To extend this transient solution technique, let $\boldsymbol{P}_i$ be the transition probability matrix of process X_i after uniformization, we have:

$$\boldsymbol{\pi}(t) = \sum_{n=0}^{\infty} \boldsymbol{\pi}_i(n) e^{-\Lambda_i t} \frac{(\Lambda_i t)^n}{n!} \qquad \text{for } t_{i-1} \leq t < t_i \quad (12)$$

$$\boldsymbol{\pi}_i(n) = \boldsymbol{\pi}_i(n-1)\boldsymbol{P}_i \quad (13)$$

$$\boldsymbol{\pi}_i(0) = \boldsymbol{\pi}(t_{i-1}) \quad (14)$$

Let ϵ be the component-wise error bound, we spread the error into all intervals (weighted) by:

$$\epsilon_i \quad = \quad \frac{t_i - t_{i-1}}{t}\epsilon \tag{15}$$

And N_i can be computed based on ϵ_i by:

$$\epsilon_i \quad \leq \quad 1 - \sum_{n=0}^{N_i} e^{-\Lambda_i t} \frac{(\Lambda_i t)^n}{n!} \tag{16}$$