Exploring Bundling Sale Strategy in Online Service Markets with Network Effects

Weijie Wu*, Richard T. B. Ma† and John C. S. Lui‡
*Shanghai Jiao Tong University †National University of Singapore ‡The Chinese University of Hong Kong
Email: weijiewu@sjtu.edu.cn, tbma@comp.nus.edu.sg, cslui@cse.cuhk.edu.hk

***Abstract*—In recent years, we have witnessed a growing trend for online service companies to offer "*bundling sales*" to increase revenue. Bundling sale means that a company groups a set of its products/services and charges this bundle at a fixed price, which is usually less than the total price of individual items. In this paper, our goal is to understand the underlying dynamics of bundling, in particular, what is the optimal bundling sale strategy and under what situations it will be more attractive than the separate sales. We focus on online service markets that exhibit *network effect*. We provide mathematical models to capture the interactions between buyers and sellers, analyze the market equilibrium and its stability, and formulate an optimization framework to determine the optimal sale strategy for the service provider. We analyze the impact of the key factors, including the network effects and operating costs, on the profitability of bundling. We show that bundling is more profitable than separate sale in most cases; however, the heterogeneity of services and the asymmetry of operating costs reduce the advantage of bundling. These findings provide important insights in designing proper sale strategies for online services.**

I. Introduction

As the economy becomes more global and competitive, it is becoming more important for online service companies to find new ways to increase revenue. One way is by *bundling services*. Bundling service (or bundling sale) means that companies group a set of their products/services and propose a single price for this group. Usually, the price of this bundling service is less than the total price of individual items. In online service markets, services are provided over the Internet infrastructure, and the service contents may vary. Typical services include instant messaging, online social networks, online games, online recommendation systems, etc. Companies usually want to expand the service scale to be as large as possible so as to increase their market share. Although most online service providers do not charge ordinary services, they do charge users for premium services, e.g., the largest movie recommendation network, IMDb, has an "IMDbPro" session where premium services ("Get informed", "Get connected" and "Get discovered") are provided to paid-users only. These premium services are often provided in a bundling manner, e.g., customers are not allowed to buy these three "Get" services separately, but they have to pay a uniform price so as to obtain the premium services as a whole. Though this is a common sale strategy in online service markets, researchers have limited understandings of the underlying rationales.

978-1-4799-3360-0/14/$31.00 © 2014 IEEE

There are a number of reasons why service providers offer bundling sales. An appealing justification for providers is cost saving. The online services usually share the same network or storage infrastructure; therefore, the cost of providing an extra service on the same infrastructure is often marginal.

Another important reason is that by bundling, the service providers can *reduce the variance* of customers' reservation prices on the services, thereby increasing the revenue of the product. In here, a customer's reservation price refers to a value such that she will purchase the service if and only if its price is no higher than this value. For example, if customer 1's reservation price on service A is \$5, and if the sale price of service A is less than or equal to \$5, then customer 1 will *subscribe* to such service. Note that different customers have different reservation prices towards each service. In Table I, we use a simple example to illustrate this concept. Suppose a company sells two services (A and B) to two customers. The second and third column depict both customers' reservation prices on the services. Assume a customer's reservation price of the bundle is the sum of reservation prices of individual services, then the two customers have the same reservation price on the bundle. If the services are sold separately, they can be priced at \$5 and attract both customers (hence the revenue is \$20); or they can be priced at \$10 and attract only one customer for each service (hence the revenue is \$20, too). In contrast, if services are bundled and priced at \$15, then both customers will purchase the service, and the total revenue is \$30. This shows that bundling can reduce the *variance* of customers' reservation prices on these services, and thus the company can increase its revenue.

	Service A	Service B	Bundle
Customer 1	\$5	\$10	\$15
Customer 2	\$10	\$5	\$15

TABLE I: An example of bundling sales

One important unique feature of online services is the *"network effect"*. It refers to the market effect at the customer's side where a particular customer's interest on a service is influenced by other customers' purchasing decisions. For example, in online social networks (e.g., Facebook, LinkedIn, Twitter, IMDb, etc.), when the number of membership increases, the benefit each member receives also increases due to a higher degree of interaction and efficiency of information spreading, and this causes more users to subscribe to this service. This is a prime example of an online service market where a large population size indicates a positive influence on each customer's valuation, and we call this the *"positive network effect"*[1]. This effect has a major impact on the choice of pricing strategies for online service providers.

A number of existing research [1], [10], [11], [13] discussed bundling strategies, but most of them focused on *non-digital goods or services*, and were mainly based on graphical explanations, case studies or algorithmic approaches. Very limited work has been focusing on formal mathematical models to provide deeper insights for the companies. Furthermore, most existing work did not consider the impact of network effect, so they can only provide limited insights on the online service market. We aim to address the following questions:

- Is it more profitable for online service providers to bundle a number of services and sell them at a single price?
- What are the factors that impact the optimal pricing strategy with network effect?

In this paper, we make the following contributions:

- We build a mathematical model which captures the online service market with network effect.
- We analyze the market equilibrium and formulate an optimization framework to capture the optimal sale strategy.
- We discuss the impact of different factors on the profitability of the bundling strategy. We show that bundling is more profitable than separate sale in most cases, while the heterogeneity of services and asymmetry of operating costs reduce the advantage of bundling.

Our paper is organized as follows. Sec. II presents a general model to capture customers' purchasing decision and the service provider's profit. Sec. III focuses on the online service market, analyzes the market equilibrium and its stability, and presents an optimization framework to capture the optimal sale strategies. In Sec. IV and Sec. V, we analyze the role of network effect and operating cost on the profitability of bundling. Sec. VI states related work and Sec. VII concludes.

II. General Model

We present a general model to characterize the Internet service market, and to study how customers and the service provider make their purchasing/pricing decisions. Let us first provide formal definitions on the sale strategies.

Definition *1:* **Separate sale** is a strategy by which a service provider sells each service S_i at price p_i individually. Customers can choose to purchase any service or not.

Definition *2:* **Bundling sale** is a strategy by which a service provider offers to sell a set of services as a single unit. The bundling service is priced at p_b. Customers can only choose to purchase the whole bundling service or not.

[1]There is also a *negative network effect* if a large number of users cause congestion. However, congestion is a physical level infrastructural problem but is not the focus of our paper on the application level pricing problem.

A. Utility functions of separate sales

Customers' utility. A customer determines whether to purchase the service(s) provided by the service provider. We consider a single service provider and a continuum of customers with different reservation prices on the service. The customers' heterogeneity in reservation prices is represented by their *types*, i.e., each infinitesimal customer is characterized by a one-dimensional type parameter $\theta \in \Theta$, which has a continuous distribution over Θ. The customer's utility function describes her purchasing behavior: a customer subscribes to a service if and only if she achieves a non-negative utility. This utility function depends on 1) the customer's reservation price on the service, and 2) the sale price p_i of the service. We assume customer θ's reservation price on S_i is $v_i(\theta)\rho_i(\delta_i)$, where $v_i(\theta)$ is her intrinsic valuation on S_i, δ_i is the fraction of customers that subscribe to S_i, and $\rho_i(\delta_i)$ is a non-decreasing function in δ_i representing the network effect. The multiplication form $v_i(\theta)\rho_i(\delta_i)$ is to characterize that a customer with a higher intrinsic valuation is more sensitive to the network effect [6], and that the reservation price is zero if $\rho_i(\theta) = 0$. We define $u_i(\theta)$ as customer θ's utility on service S_i:

$$u_i(\theta) = v_i(\theta)\rho_i(\delta_i) - p_i. \tag{1}$$

Customers of different types have different intrinsic valuations $v_i(\theta)$ on S_i, and we assume $v_i(\theta)$ has a continuous distribution in θ over Θ. We further denote $f(\theta)$ as the density function of θ, and define

$$H_i(x) \triangleq \int_{\theta\in\Theta} \mathbf{1}_{\{v_i(\theta)\le x\}} f(\theta) d\theta \tag{2}$$

as the cumulative distribution function of $v_i(\theta)$, i.e., given any value x, $H_i(x)$ represents the fraction of customers whose intrinsic valuation on service S_i is less than or equal to x.

Service provider's utility. The service provider determines whether it should provide separate or bundling sale, and to propose the price(s) for the service(s). We model the service provider's utility as its total profit, and we will use "utility" and "profit" interchangeably in later analysis. We consider two factors that impact the service provider's utility: 1) the service fee received from customers, which we model as $p_i\delta_i$; 2) the variable operating cost[2] which we model as $m_i\delta_i$. In here, m_i represents the *per-unit variable cost*[3], and we call it the *unit operating cost* or the *unit cost* for short. We define

$$U_i = (p_i - m_i)\delta_i \tag{3}$$

as the service provider's utility on service S_i. Therefore, the service provider's utility from all separate sale services is

$$U_s = \sum_{i=1}^{I}(p_i - m_i)\delta_i. \tag{4}$$

[2]Variable cost and fixed cost consist of the total cost. We consider the variable cost only since the fixed cost only represents a linear shift on the utility and does not affect our conclusion.

[3]Some existing literature uses the term "marginal cost" to represent this concept. In fact, if the marginal cost is a constant, its value is equal to the per-unit variable cost which we define here.

B. Utility functions of the bundling sale

The previous subsection has set up the stage for expressing the utilities of customers and the service provider when services are sold *individually*. We now consider the bundling strategy which bundles services $S_1, S_2, \ldots, S_I$. Often times, customers view the bundled services as a whole. We use the notation b to denote the bundling service S_b. For consistency, we still assume that the network effect function impacts the utilities of the bundle in a multiplication form. By substituting b for i we denote the corresponding notations for S_b. In particular, $u_b(\theta)$, $v_b(\theta)$, δ_b and p_b represent customer θ's utility on purchasing S_b, her intrinsic valuation on S_b, the fraction of users purchasing S_b, and the price charged for S_b, respectively. We have the customer's and the service provider's utility functions as

$$u_b(\theta) = v_b(\theta)\rho_b(\delta_b) - p_b \text{ and } U_b = (p_b - m_b)\delta_b, \qquad (5)$$

where m_b denotes the unit cost for S_b. In particular, we assume $m_b = \sum_{i=1}^{I} \beta_i m_i$ where $\beta_i \in [0,1]$ denotes the *scaling factors* of the operating cost. In fact, $\beta_i \leq 1$ implies that bundling can save unit costs, e.g., if we bundle a number of bandwidth-related functionalities in online game services, then the services can rely on the same infrastructure and save cost.

C. Market equilibrium

Due to the network effect, and that customers subscribe to services at different times, the above model is in fact a dynamic system. We use the following definition to describe the steady state of the system.

Definition *3:* Given price p_i, $\delta_i > 0$ is a *market equilibrium* if

$$\int_{\theta\in\Theta} \mathbf{1}_{\{v_i(\theta)\rho_i(\delta_i)\geq p_i\}} f(\theta) d\delta = \delta_i, \qquad (6)$$

where $f(\theta)$ is the density function of θ.

This definition states that for any given customer's utility as $u_i(\theta) = v_i(\theta)\rho_i(\delta_i) - p_i$, if exactly δ_i fraction of customers have a non-negative utility to purchase S_i, then δ_i is a market equilibrium. It represents the fraction of customers who purchase the service S_i when the system reaches a steady state, i.e., given this fraction, no customer has an incentive to change her decision. In the following, our analysis is based on this equilibrium. We will discuss pricing strategies under such scenario. Unless we state otherwise, we will use δ_i to denote the equilibrium in the remaining of this paper.

Note that when $\delta_i = 0$, it may also be a steady state with no users. But in this case, the service is closed and there is no real market. Thus, we exclude $\delta_i = 0$ from the definition of equilibrium. Now let us characterize the value of δ_i.

Lemma *1:* Assume $\rho_i(\delta_i) > 0$ for any $\delta_i > 0$. The value δ_i is an equilibrium if and only if it satisfies $\delta_i = 1 - H_i\left(\frac{p_i}{\rho_i(\delta_i)}\right)$, where $H_i(\cdot)$ is the cumulative distribution function of $v_i(\theta)$.

Proof: Please refer to the appendix. ∎

The above lemma gives an implicit form to characterize and compute the equilibrium. In later analysis, it is more convenient to use the following corollary.

Corollary *1:* Assume $H_i(\cdot)$ is a strictly increasing function in $[0, V_i]$, and $\rho_i(\delta_i) > 0$ for any $\delta_i > 0$. Given any price p_i, if there exists an equilibrium δ_i, then it is a solution to the following equation:

$$p_i = \rho_i(\delta_i) H_i^{-1}(1 - \delta_i), \qquad (7)$$

where $H_i^{-1}(\cdot)$ is the inverse function of $H_i(\cdot)$ defined in $[0, 1]$.

Up till now we have set up a general model to capture the customers' and the service provider's utilities. Based on this model, we will proceed to analyze the properties of the market.

III. Online Service Market: Equilibrium and Optimal Sale Strategy

In this section we study an online service market. We first model the users' valuation distribution and the network effect, and then analyze the market equilibrium. Lastly, we establish a framework to determine the optimal sale strategies.

A. Distributions of users' intrinsic valuations

Let us first state two basic assumptions on the users' intrinsic valuation distributions. First, given a customer θ's intrinsic valuation on each individual service S_i as $v_i(\theta)$, we assume that her valuation on the bundle satisfies $v_b(\theta) \geq \sum_{i=1}^{I} v_i(\theta)$. The rationale is that the online services are often complementary, i.e., using them in conjunction can give customers *extra* utilities[4]. Second, we assume the customers' intrinsic valuation on different services are independent. Therefore, if we let $H_i(x)$ and $H_B(x)$ be the cumulative distribution functions of $v_i(\theta)$ and $\sum_{i=1}^{I} v_i(\theta)$, respectively, we have $H_B(x) = H_1(x) \otimes H_2(x) \otimes \cdots \otimes H_I(x)$, where the convolution operation is defined by $H_i(x) \otimes H_j(x) = \int H_i(x-t) dH_j(t)$.[5]

Let $H_b(x)$ be the cumulative distribution of $v_b(\theta)$. Since $v_b(\theta)$ is *lower bounded* by $\sum_{i=1}^{I} v_i(\theta)$, i.e., any customer's intrinsic valuation on the bundle is *at least* $\sum_{i=1}^{I} v_i(\theta)$, thus, $H_b(x)$ is *upper bounded* by $H_B(x)$, i.e., given any value x, the fraction of users whose $v_b(\theta) \leq x$ is *at most* $H_B(x)$. In fact, $H_b(x)$ equals $H_B(x)$ if $v_b(\theta) = \sum_{i=1}^{I} v_i(\theta)$. In what follows, we use $H_B(x)$ for the baseline analysis. It is easy to see that if in this baseline analysis, bundling achieves a profit gain over the separate sale under a certain circumstance, it must achieve an equally high or even higher profit gain under the distribution $H_b(x)$.

We focus on bundling two services S_1 and S_2. This model represents a wide range of bundling strategy decisions, since any bundling of multiple services can be consequentially constructed by bundling two services. We consider the uniform distribution[6] of customer's intrinsic valuation, which is widely

[4] For example, in the IMDb premium services, "Get informed" can help users to fully utilize the "Get connected" and "Get discovered" functionalities.

[5] This is a standard result in probability theory and we omit its proof.

[6] We use the uniform distribution for ease of mathematical derivation. However, the underlying rationale of bundling does not rely on this specific form, so our framework and insights do hold for other distributions.

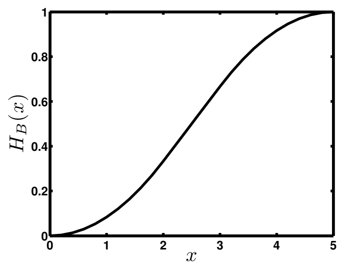


Fig. 1: Cumulative distribution $H_B(x), V_1 = 2, V_2 = 3$

adopted in economic literatures (e.g., [2], [10]). Formally, we have the cumulative distribution of $v_i(\theta), i = 1, 2$ as

$$H_i(x) = \begin{cases} 0 & \text{if } x < 0, \\ x/V_i & \text{if } 0 \le x \le V_i, \\ 1 & \text{if } x > V_i, \end{cases} \tag{8}$$

where $V_i (i = 1, 2)$ is the maximal intrinsic valuation of S_i. Without loss of generality, we let $V_1 \le V_2$, and we have

Lemma *2:* The baseline distribution function $H_B(x)$ is

$$H_B(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2/(2V_1V_2) & \text{if } 0 \le x \le V_1, \\ (2x - V_1)/(2V_2) & \text{if } V_1 < x \le V_2, \\ 1-(V_1+V_2-x)^2/(2V_1V_2) & \text{if } V_2 < x \le V_1+V_2, \\ 1 & \text{if } x > V_1 + V_2. \end{cases}$$

Proof: By taking the convolution operations on $H_i(x)$ we can directly reach the result. ∎

In Fig. 1, we illustrate the shape of this distribution. It shows that $H_B(x)$ increases more rapidly in the middle range of the interval; in other words, customers are more *concentrated* to have a moderate valuation of the bundle, comparing with the uniform distribution of separate services. This shows that bundling can *reduce the variance* of customers' valuations, and it is an important underlying reason to make bundling profitable: if the service provider sets a relatively low bundling price, then it becomes easier for him to attract more customers since there are a lot of customers with moderate valuations, and hence the service provider can make more profit. However, if the service provider only targets at a small amount of customers with high valuations, then bundling may not have an advantage. This is because $H_B(x)$ indicates fewer customers with high valuations comparing to the uniform distribution.

B. Network effect and utility functions

We model the network effect in form of $\rho_i(\delta_i) = \delta_i^{\alpha_i}$, where $\alpha_i \in (0, +\infty)$ represents the shape of the network effect. When the two services with $\alpha_1 \le \alpha_2$ are bundled, customers view it as a new service with the network effect function $\rho_b(\delta_b) = \delta_b^{\alpha_b}$, where $\alpha_1 \le \alpha_b \le \alpha_2$.

We use the above form for a number of reasons. First, $u_i(\theta) = 0$ if $\delta_i = 0$, i.e., no customer has an incentive to enter an empty market. This is a common fact in many interactive applications, e.g., online social network or recommendation systems; and this also shows that it is important for a service provider to promote the service and have some initial users for startup. Second, $\delta_i^{\alpha_i}$ is increasing in δ_i, so it represents a positive network effect. Last but not least, this is an *iso-elasticity* function which allows us to use a single parameter α_i to model the elasticity, or the shape of the network effect. Large α_i (or a *convex* function) means that given a small δ_i, $\delta_i^{\alpha_i}$ is small and many users will lose their interest, so a large start-up population is necessary. On the contrary, if α_i is small (or a *concave* function), then $\delta_i^{\alpha_i}$ is large given a moderate or small δ_i. This means a small amount of initial users can be enough to induce a large network effect, and later on, the service can potentially attract many more customers. Note that our model generalizes the linear network effect models in many existing literatures [6], [12]; in fact, when $\alpha_i = 1$, our model exactly represents the linear network effect.

Based on the above settings, service S_i is uniquely defined by 1) the users' maximal intrinsic valuation V_i, 2) the unit operating cost m_i, and 3) the network effect parameter α_i. In later analysis, we use a tuple $S_i = \langle V_i, m_i, \alpha_i \rangle$, $i = 1, 2$ to denote a separate sale service. Based on the above, customer θ's utility and the service provider's utility on service S_i are:

$$u_i(\theta) = v_i(\theta)\delta_i^{\alpha_i} - p_i, \quad U_i = (p_i - m_i)\delta_i; \tag{9}$$

and the service provider's utility on all separate sales is

$$U_s = \sum_{i=1}^{2} U_i = \sum_{i=1}^{2} (p_i - m_i)\delta_i. \tag{10}$$

For the bundling sale, the utility functions are

$$u_b(\theta) = v_b(\theta)\delta_b^{\alpha_b} - p_b, \quad U_b = \left(p_b - \sum_{i=1}^{2} \beta_i m_i\right)\delta_b. \tag{11}$$

C. Analysis of market equilibrium

In this subsection, we derive the conditions for the existence of the market equilibrium (or equilibria).

Theorem *1:* Consider any separate or bundling sale $S_i, i \in \{1, 2, b\}$. There exists a threshold $\bar{p_i}$, such that for any given service price p_i, we have

$$\text{\# of equilibrium (or equilibria)} = \begin{cases} 0 & \text{if } p_i > \bar{p_i}, \\ 1 & \text{if } p_i = 0 \text{ or } \bar{p_i}, \\ 2 & \text{if } p_i \in (0, \bar{p_i}). \end{cases}$$

In particular, for separate sale, we have $\bar{p_i} = \frac{V_i}{\alpha_i+1}\left(\frac{\alpha_i}{\alpha_i+1}\right)^{\alpha_i}$.

Proof: Please refer to the appendix. ∎

This theorem states that the condition for the existence of equilibrium is that the service price is not too high, otherwise no customer will purchase the service. We also note that if the existence is guaranteed, then almost surely there are two equilibria. We next discuss the stability property and explain why we are interested in the larger equilibrium.

Discussion on stability[7]**.** We say that an equilibrium δ is *stable*, if there exists a positive ϵ, such that if at any time, a non-equilibrium fraction $\delta' \in (\delta - \epsilon, \delta + \epsilon)$ of customers subscribe to the service, then the dynamic market will eventually reach the equilibrium δ. In fact, if we consider the two equilibia $\delta_i^1 < \delta_i^2$ in the above theorem, the only *stable* one is δ_i^2. If

[7]The features of equilibria are quite similar to discussions in [4].

the market is now with $\delta_i^1 - \epsilon$ fraction of customers, then eventually all customers will leave the market and the service will be closed; if the market is now with $\delta_i^1 + \epsilon$ fraction of customers, it will not reach δ_i^1 but will reach δ_i^2. Hence, δ_i^1 is an *unstable* equilibrium. In contrast, if we consider the market with any fraction $\delta_i^{2'} \in (\delta_i^2 - \epsilon, \delta_i^2 + \epsilon)$ of users, the dynamic market will eventually reach the equilibrium δ_i^2; hence, δ_i^2 is a *stable* equilibrium. Due to the page limit, we omit the details here; interested readers may refer to [6] for a detailed discussion. Due to this stability property, in later analysis, we can safely restrict our analysis in the larger equilibrium. We use $\max\{\delta_i(p_i)\}$ to denote the maximal equilibrium for a given p_i, and define $\max\emptyset = 0$ to capture an empty market when the price is too high and the equilibrium does not exist.

D. An optimization framework for the sale strategies

In this section we establish an optimization framework to determine the optimal sale strategies. A natural way to model the optimal sales is that the service provider aims to find a price (or prices) for the separate or the bundling sale which maximizes its profit, i.e., the optimal separate or bundling sale $S_i, i \in \{1, 2, b\}$, can be modeled as

$$\begin{aligned} \max_{p_i} \quad & U_i(p_i) = (p_i - m_i)\max\{\delta_i(p_i)\}, \\ \text{s.t.} \quad & p_i \geq 0. \end{aligned} \tag{12}$$

However, this form is not easy for analysis, and we opt to change the decision variable from p_i to δ_i. According to Corollary 1, we can transform (12) into the following problem[8]

$$\begin{aligned} \max_{\delta_i} \quad & U_i(\delta_i) = (\rho_i(\delta_i)H_i^{-1}(1-\delta_i) - m_i)\delta_i, \\ \text{s.t.} \quad & 0 \leq \delta_i \leq 1. \end{aligned} \tag{13}$$

Since the above optimizations for $i \in \{1, 2, b\}$ have continuous objective functions over a compact set, they are guaranteed to have optimal solutions. By solving the above optimizations, we automatically choose the largest equilibrium fraction for any given price, which is the stable one as we desire.[9] Up till now we have established an optimization framework to determine the optimal sale strategies. In what follows, we use U_s^* and U_b^* to denote the maximal profit of the service provider, under the separate and bundling sales respectively. If we could calculate U_s^* and U_b^*, then we can determine whether bundling is more profitable than separate sale by comparing their values. Formally, we have the following definition to capture the profit gain of bundling over the separate sale:

Definition *4:* The *profit gain ratio* γ is the difference between the maximal profit of the bundling sale and that of the separate sale, divided by maximal profit of the separate sale, i.e.,

$$\gamma = (U_b^* - U_s^*)/U_s^*. \tag{14}$$

If $\gamma > 0$, it means the optimal bundling sale is more profitable than the optimal separate sale, and vice versa. A larger value of γ indicates a larger profit gain by the bundling sale.

[8]Although $\delta_i = 0$ is excluded from Corollary 1, we can verify that if the solution to (12) is p_i^* and $\max\{\delta_i(p_i^*)\} = 0$, then the solution to (13) is 0.

[9]To see this, note if there exist $\delta_i^1 < \delta_i^2$ such that $\rho_i(\delta_i^1)H_i^{-1}(1-\delta_i^1) = \rho_i(\delta_i^2)H_i^{-1}(1-\delta_i^2) \geq m_i$, then obviously δ_i^1 cannot be the solution to (13).

We believe this framework is critical for online service providers to evaluate their best sale strategies; but we should also point out it may not be easy to have general results at this stage, in particular, when a general α_i induces difficulty in solving the optimizations. In the following sections, we explore the impact of various factors, i.e., the network effect parameter α_i, and the unit cost m_i, on the profitability of bundling. Before we proceed, let us present the following lemma which reflects the scaling properties of the sales.

Lemma *3 (Scaling property):* Let c be a positive number.
(1) If the equilibrium and profit of the optimal sale $S_i = \langle V_i, m_i, \alpha_i\rangle$ are δ_i^* and U_i^*, then the equilibrium and profit of the optimal sale $S_i' = \langle cV_i, cm_i, \alpha_i\rangle$ are δ_i^* and cU_i^*.
(2) If the profit gain ratio for bundling $S_1 = \langle V_1, m_1, \alpha_1\rangle$ and $S_2 = \langle V_2, m_2, \alpha_2\rangle$ is γ , then the profit gain ratio for bundling $S_1' = \langle cV_1, cm_1, \alpha_1\rangle$ and $S_2' = \langle cV_2, cm_2, \alpha_2\rangle$ is also γ.

Applying the optimization framework we can easily prove the above lemma. This lemma points out that if V_i and m_i increases (or decreases) by the same factor, then it does not impact the equilibrium or the profitability of bundling. Hence, in later analysis, we can normalize V_1 to be 1 and vary V_2 so as to explore the whole design space. This simplifies our analysis and does not lose any generality.

IV. Impact of Network Effect

Up to now we have formulated a framework to capture the pricing strategies and the market equilibrium. In this and the next sections we discuss the impact of key factors on the profitability of bundling. We first focus on the network effect.

Many online service providers incur a much larger fixed cost comparing to their variable cost. For example, online social network service needs to invest a large amount of money to initially set up the hardware and infrastructure, but the cost is minimal to increase one user membership. So in this section, we set the per-unit variable cost $m_i = 0$ and consider $S_i = \langle V_i, 0, \alpha_i\rangle$. The service provider's utility can be expressed as

$$U_s = p_1\delta_1 + p_2\delta_2. \tag{15}$$

This simplification captures the feature of a wide range of digital online services, and it allows us to isolate different factors so as to better understand the impact of network effect.

A. Homogeneous network effect ($\alpha_1 = \alpha_2 = \alpha$)

We start our discussion with the network effect functions $\rho_1(\delta) = \rho_2(\delta) = \delta^\alpha$. Naturally, we assume the bundling service S_b also have $\rho_b(\delta) = \delta^\alpha$. Such setting represents bundling two services with similar network effects. We have the following theorem to show that bundling is more profitable than separate sales under this setting.

Theorem *2:* Consider $S_1 = \langle V_1, 0, \alpha\rangle$ and $S_2 = \langle V_2, 0, \alpha\rangle$.
1) The profit gain ratio of the bundling sale $\gamma > 0$.
2) In particular, when $S_1 = S_2 = \langle V, 0, \alpha\rangle$, we have
  a) The optimal separate sale is

$$\delta_i^* = \frac{\alpha+1}{\alpha+2},\ p_i^* = \frac{V}{\alpha+2}\left(\frac{\alpha+1}{\alpha+2}\right)^\alpha,\ U_s^* = 2\delta_i^*p_i^*.$$

  b) The optimal bundling sale is

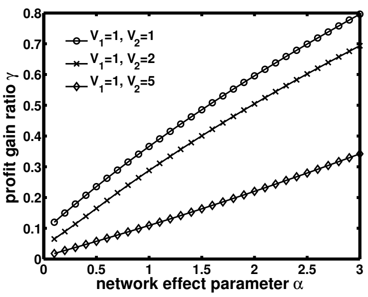
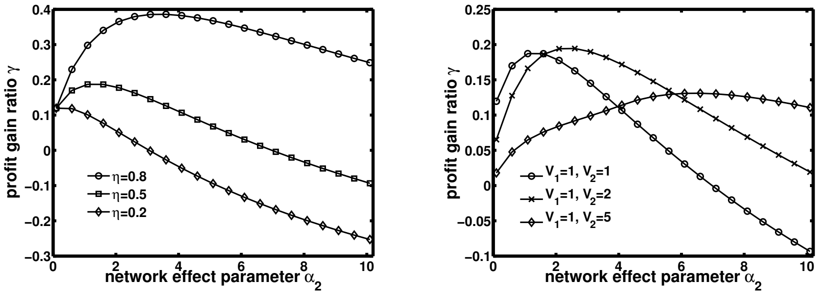


Fig. 2: Impact of network effect, $\alpha_1 = \alpha_2$



(a) $V_1 = V_2$

(b) $V_1 \neq V_2$

Fig. 3: Impact of network effect, $\alpha_1 \neq \alpha_2$

$$\delta_b^* = \frac{2\alpha+2}{2\alpha+3},\ p_b^* = \left(\frac{2\alpha+2}{2\alpha+3}\right)^{\alpha}\sqrt{\frac{2}{2\alpha+3}}V,\ U_b^* = \delta_b^* p_b^*.$$

c) The profit gain ratio of the bundling sale is

$$\gamma(\alpha) = \frac{\sqrt{2}}{4}\frac{(2\alpha+4)^{\alpha+2}}{(2\alpha+3)^{\alpha+3/2}} - 1,$$

and it is an increasing function in α.

Proof: Please refer to the appendix. ∎

This theorem states that when $\alpha_1 = \alpha_2$, bundling is *always more profitable*, and *large* α (i.e., a *convex* network effect function) indicates a *high* profit gain. Let us use examples to show how bundling achieves a higher profit. In Fig. 2 we consider $S_1 = \langle 1, 0, \alpha\rangle$ and $S_2 = \langle V_2, 0, \alpha\rangle$, where we vary $V_2 \in \{1, 2, 5\}$ and $\alpha \in [0.1, 3.0]$. We can see γ is always positive, and this validates our results in Theorem 2. We can also observe that when there is a large gap between V_1 and V_2, the profit gain ratio γ reduces. This is because the joint distribution $H_B(x)$ becomes less concentrated in the middle range, so bundling sale can attract fewer customers. To summarize, we have the following observation:

Observation 1: The advantage of bundling becomes more dominant when the network effect function is more convex (i.e., larger α); however, the heterogeneity in intrinsic valuation distributions reduces the profit gain ratio of bundling.

B. Heterogeneous network effect $(\alpha_1 \neq \alpha_2)$

Now let us consider bundling two services with different network effects. We first consider $V_1 = V_2$. It is natural to assume α_b, the network effect parameter for the bundle, is between α_1 and α_2. Let $\alpha_b = \eta\alpha_1 + (1-\eta)\alpha_2$ where $\eta \in [0, 1]$. We have the following result.

Theorem *3:* Consider $S_1 = \langle V, 0, \alpha_1\rangle$ and $S_2 = \langle V, 0, \alpha_2\rangle$. Let $\alpha_b = \eta\alpha_1 + (1-\eta)\alpha_2$. The profit gain ratio is

$$\gamma = \frac{\left(\frac{2\alpha_b+2}{2\alpha_b+3}\right)^{\alpha_b+1}\sqrt{\frac{2}{2\alpha_b+3}}}{\frac{1}{\alpha_1+2}\left(\frac{\alpha_1+1}{\alpha_1+2}\right)^{\alpha_1+1} + \frac{1}{\alpha_2+2}\left(\frac{\alpha_2+1}{\alpha_2+2}\right)^{\alpha_2+1}} - 1. \quad (16)$$

Proof: By noting the form of U_s^* and U_b^* in Theorem 2, we can directly reach the conclusion . ∎

Let us show the impact of network effect on γ. In Fig. 3(a) we consider $S_1 = \langle 1, 0, 0.1\rangle$ and $S_2 = \langle 1, 0, \alpha_2\rangle$ where we vary $\alpha_2 \in [0.1, 10.0]$. We plot three curves of γ when $\eta = 0.8, 0.5$ and 0.2, respectively. We first focus on a particular curve, e.g., $\eta = 0.5$. When α_2 increases from 0.1, γ also increases; this is because a large network effect parameter has a positive impact on bundling. But when α_2 is large, γ begins to decrease and eventually becomes negative; this is because when α_1 and α_2 differ a lot, the optimal equilibria, δ_1^* and δ_2^*, also differ a lot. In such cases it is not rational to bundle S_1 and S_2, since the bundling sale needs to find a unique equilibrium δ_b^*, which is either far away from δ_1^* or far away from δ_2^*, so bundling is not as profitable as separate sale. We also observe when η is larger, γ is also larger. This is because larger η implies larger α_b, or a more convex network effect function, which makes bundling even more profitable.

Similar to the previous discussions, we also consider the services with different V_i. In Fig. 3(b), we consider $S_1 = \langle 1, 0, 0.1\rangle$ and $S_2 = \langle V_2, 0, \alpha_2\rangle$ where we vary $\alpha_2 \in [0.1, 10]$ and $V_2 \in \{1, 2, 5\}$. We set $\eta = \frac{V_1}{V_1+V_2}$ to represent the relative weight of each service. We can observe the similar feature as we have shown before: when α_2 increases, γ first increases and then decreases. We also observe that the inflection point increases when V_2 increases. This is because when V_2 is large, service S_2 has a major impact on the bundle, so the positive impact of α_2 on bundling can be effective in a larger range. To summarize, we have the following observation:

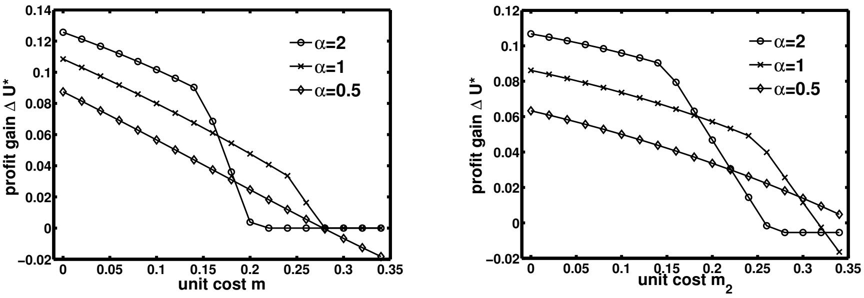
Observation 2: The heterogeneity of network effect functions decreases the profitability of bundling.

V. Impact of Operating Cost

In the previous section we have discussed the impact of network effect when the variable operating cost equals zero. Although this approximation applies to many existing services, there might be exceptions. For example, in online storage systems (e.g., Dropbox), the unit cost of storing the data might not be negligible. In this section, we discuss how the operating cost impacts the pricing strategies. Our discussions generalize the results we obtained in the previous section.

A. Impact of operating cost when $\alpha_1 = \alpha_2 = \alpha$

We start our discussion when both services have the same network effect function, i.e., $\rho_i(\delta_i) = \delta_i^{\alpha}, i = 1, 2$. We want to explore how our results in Theorem 2 can be generalized with non-zero unit operating costs. According to Lemma 3, we



(a) $m_1 = m_2$ (b) $m_1 \neq m_2$

Fig. 4: Impact of unit cost, $\alpha_1 = \alpha_2$

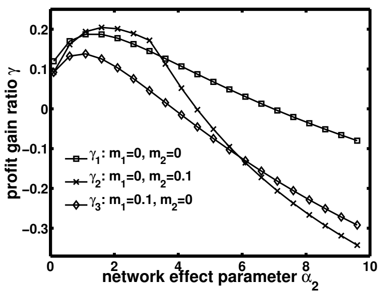


Fig. 5: Impact of unit cost, $\alpha_1 \neq \alpha_2$

can normalize V_i so that the effectiveness of the unit cost is represented by $\frac{m_i}{V_i}$. We will discuss when $m_1 : m_2 = V_1 : V_2$ and $m_1 : m_2 \neq V_1 : V_2$, i.e., symmetric and asymmetric unit costs, respectively. We start from the symmetric case.

Theorem *4:* If $\alpha \geq 1$ and $m_1 : m_2 = V_1 : V_2$, then $U_b^* \geq U_s^*$.

Proof: Please refer to the appendix. ∎

This theorem states that if S_1 and S_2 have the same convex network effect function and symmetric unit costs, then the profit of the optimal bundling is no less than that of the optimal separate sale. The key reason is that under this setting, the optimal separate sale always obtains an equilibrium larger than $1/2$, so bundling can attract more customers. However, if $\alpha_i < 1$ and m is large, then δ_i^* may be less than 1/2 and bundling may not be always profitable. Let us use examples to show this phenomenon. Since U_s^* may be zero, which leads $\gamma = \infty$, we opt to use the *profit gain*, defined by $\Delta U^* = U_b^* - U_s^*$, as the performance measure. In Fig. 4(a), we consider $S_1 = S_2 = \langle 1, m, \alpha \rangle$ and vary $m \in [0, 0.34]$, $\alpha \in \{0.5, 1, 2\}$. We observe for any given α, ΔU^* reduces with respect to m. This indicates that unit cost *reduces* the profit gain of bundling. When $\alpha = 1$ or 2, bundling is *always no worse* than separate sale. This validates our result in Theorem 4. When $\alpha = 0.5$, ΔU^* can be negative when m is large. This indicates when the network effect functions are concave, a large unit cost can make bundling less profitable than separate sale.

Now let us consider the impact of asymmetric unit costs, i.e., $m_1 : m_2 \neq V_1 : V_2$. The asymmetry induces different equilibria, and bundling is not always more profitable. It is not easy to quantify the dominant domain of the bundling or the separate sales. We have the following theorem as a sufficient condition to guarantee the profitability of bundling.

Theorem *5:* Let $\delta_i^*, i = 1, 2$ be equilibria of optimal separates sales S_1 and S_2. If $\frac{\beta_2}{2-\beta_2}\delta_1^* \leq \delta_2^* \leq \delta_1^*$ (where β_2 is the scaling factor such that $m_b = \beta_1 m_1 + \beta_2 m_2$), then $U_b^* > U_s^*$.

Proof: Please refer to the appendix. ∎

This theorem states that if the optimal equilibria of separate sales are close, then bundling is more profitable. The underlying reason is similar to the previous analysis: if two services are highly asymmetric and have very different equilibria, then it is not feasible to find a suitable service price for the bundle, because the corresponding equilibrium δ_b of the bundle, is either too far from δ_1^* or too far from δ_2^*; only when δ_1^* and δ_2^* are close, bundling can be more profitable. Let us use examples to show how the asymmetry impacts the profit gain. In Fig. 4(b), we consider $S_1 = \langle 1, 0.14, \alpha \rangle, S_2 = \langle 1, m_2, \alpha \rangle$ and vary $m_2 \in [0, 0.34]$ and $\alpha \in \{0.5, 1, 2\}$. We can observe that when m_2 increases, the profit gain ΔU^* decreases; when m_2 is greatly larger than m_1, then ΔU^* can be negative for $\alpha = 1$ or 2. Comparing with Fig. 4(a) where ΔU^* is always non-negative for $\alpha = 1$ or 2, we can see the asymmetry in unit costs further reduces the profitability of bundling.

To summarize, we have the following observation.

Observation 3. Under the symmetric operating costs and homogenous network effect, the operating costs reduce the profitability of bundling; in particular, when the network effect function is concave, bundling may be less profitable than separate sales. When the operating costs are asymmetric, the profitability of bundling is further reduced.

B. Impact of operating cost when $\alpha_1 \neq \alpha_2$

In this section, we observe the impact of the unit operating cost when the two services have different network effect functions. In particular, we show how our result in Fig. 3(a) changes with consideration of the unit cost. To keep consistency with Fig. 3(a), we evaluate the profit gain ratio γ. In Fig. 5, we consider bundling $S_1 = \langle 1, m_1, 0.1 \rangle$ and $S_2 = \langle 1, m_2, \alpha_2 \rangle$. We fix $\beta_1 = \beta_2 = 1, \eta = 0.5$, and vary $\alpha_2 \in [0.1, 10.0]$. We consider three cases of γ: $\gamma = \gamma_1$ if $m_1 = m_2 = 0$, $\gamma = \gamma_2$ if $m_1 = 0, m_2 = 0.1$, and $\gamma = \gamma_3$ if $m_1 = 0.1, m_2 = 0$. We first note $\gamma_3 < \gamma_1$. This means a unit cost on S_1 discourages bundling, which is the same as our finding in the previous subsection. Next we focus on the curve of γ_2, and we have some interesting observations. We note $\gamma_2 > \gamma_1$ when α_2 is moderately large. This shows the unit cost of S_2 can sometimes increase the profitability of bundling. The reason is this unit cost reduces δ_2^*, so that the gap between δ_1^* and δ_2^* reduces. Therefore, the unit cost reduces the asymmetry of S_1 and S_2, so it increases the profitability of bundling. However, when α_2 is quite large, the unit cost further reduces δ_2^* and its negative impact on the profit becomes dominant. To summarize, we have the following observation:

Observation 4. The operating costs play a complicated role when network effect functions are different. In particular, a moderate operating cost on the service with larger α_i may increase the profitability of bundling.

VI. Related Work

Bundling strategy has been discussed in economic community. Early studies [1], [10], [11] revealed basic understandings and they were all based on non-digital goods. Authors in [2], [3], [8] discussed bundling strategy of digital goods with zero marginal cost but there was no network effect. Network effect (or network externality) has also been extensively studied. Early works [7], [9] set up basic models to define and analyze network effect, and recent works [5], [14], [15] have discussed various applications under network effects.

Although network effect and bundling sale have been both extensively studied, there are very few works that combine them. We only find one recent work [12] closely related to our paper, where the authors discussed bundling strategy of technological products with network externality. The paper presented interesting findings, but they mainly rely on numerical and graphical explanations, and their analysis was restricted to some special cases. Meanwhile, the linear and additive form of network externality applied in their work is in a special form which does not capture all important features of online services. Hence, we need a more accurate model on the network effect to capture today's online market. Our paper differs from previous works in that 1) we build a formal optimization framework that captures the optimal separate sale and bundling strategies, 2) we give rigorous analytical results based on the multiplication form of network effect, and 3) we analytically show how network effects and operating costs impact on the profitability of bundling under various scenarios.

VII. Conclusion

In this paper, we discuss the bundling sale strategy for online service markets which exhibit network effects. In such market, a customer's purchasing decision is influenced by other customers' purchasing decisions. We formulate a formal optimization framework to characterize the optimal sale strategies, which allows the service providers to determine their best sale strategies. Based on this, we analyze and quantify the impact of the key factors. Our important findings include: 1) when the network effect function is more convex, the profit gain of bundling over separate sales becomes larger; 2) the operating cost usually plays a negative role towards bundling; but when the two services have different network effects, a moderate operating cost on a particular service may increase the profitability of bundling; and 3) the asymmetry in operating costs, and the heterogeneity in valuation distributions or network effects, reduce the profitability of bundling, and can even make bundling less profitable than separate sales. We believe these findings provide valuable insights for online service providers to design effective pricing schemes, and we plan to better explore bundling sales via real data analytics.

Acknowledgement: Weijie Wu is supported in part by National Key Basic Research Program of China (NKBRP) 2013CB329603, and National Natural Science Foundation of China Project 61171173. Richard T.B. Ma is supported in part by the Ministry of Education of Singapore under AcRF Grant R-252-000-526-112. John C.S. Lui is supported in part by the RGC Grant (415112).

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Appendix

Proof of Lemma 1: Since for any $\delta_i > 0$ we have $\rho_i(\delta_i) > 0$, so the condition $v_i(\theta)\rho_i(\delta_i) \geq p_i$, is equivalent to $v_i(\theta) \geq \frac{p_i}{\rho_i(\delta_i)}$. According to the definition of equilibrium, we have $\delta_i = \int_{\theta\in\Theta} \mathbf{1}_{\left\{v_i(\theta)\geq\frac{p_i}{\rho_i(\delta_i)}\right\}} f(\theta)d\theta$. By noting the above equation and recalling the definition of $H_i(x)$ in Eq. (2), we can prove the lemma. ∎

Proof of Theorem 1: According to Corollary 1, δ_i is an equilibria if and only if it satisfies

$$p_i = \delta_i^{\alpha_i} H_i^{-1}(1-\delta_i),\ (i=1,2),\ \ p_b = \delta_b^{\alpha_b} H_B^{-1}(1-\delta_b),$$

where $H_i^{-1}(1-\delta_i) = V_i(1-\delta_i)$ and

$$H_B^{-1}(1-\delta_b)=\begin{cases} V_1+V_2-\sqrt{2V_1V_2\delta_b} & \text{if } 0\leq\delta_b\leq\frac{V_1}{2V_2}, \\ \frac{1}{2}V_2+V_2(1-\delta_b) & \text{if } \frac{V_1}{2V_2}<\delta_b\leq 1-\frac{V_1}{2V_2}, \\ \sqrt{2V_1V_2(1-\delta_b)} & \text{if } 1-\frac{V_1}{2V_2}<\delta_b\leq 1. \end{cases}$$

Let $g_i(x) = x^{\alpha_i}H_i^{-1}(1-x), i = 1,2$ and $g_b(x) = x^{\alpha_b}H_B^{-1}(1-x)$. Then equilibrium δ_i is a solution to

$$p_i = g_i(\delta_i). \tag{17}$$

By letting $g_i'(x)=0, i=1,2$, we see there is a unique solution $x_i^*=\frac{\alpha_i}{\alpha_i+1}$ in $(0,1]$. By letting $g_b'(x)=0$ we have

$$x_b^*=\begin{cases}\frac{2\alpha_b^2(V_1+V_2)^2}{(4\alpha_b^2+4\alpha_b+1)V_1V_2} & \text{if } 0\le x\le\frac{V_1}{2V_2},\\ \frac{\alpha_b(V_1+2V_2)}{2(\alpha_b+1)V_2} & \text{if } \frac{V_1}{2V_2}<x\le 1-\frac{V_1}{2V_2},\\ \frac{2\alpha_b}{2\alpha_b+1} & \text{if } 1-\frac{V_1}{2V_2}<x\le 1.\end{cases}$$

Since $\frac{\alpha_b}{2\alpha_b+1}<\frac{1}{2}<1-\frac{V_1}{2V_2}$, it is not a solution to $g_b'(x)=0$. If $\frac{\alpha_b(V_1+2V_2)}{(2\alpha_b+1)V_2}>\frac{V_1}{2V_2}$, then $\frac{V_2}{V_1}>\frac{1}{2\alpha_b}$. If $\frac{2\alpha_b^2(V_1+V_2)^2}{(4\alpha_b^2+4\alpha_b+1)V_1V_2}\le\frac{V_1}{2V_2}$, then $\frac{V_2}{V_1}\le\frac{1}{2\alpha_b}$. This means $\frac{\alpha_b(V_1+2V_2)}{(2\alpha_b+1)V_2}$ and $\frac{2\alpha_b^2(V_1+V_2)^2}{(4\alpha_b^2+4\alpha_b+1)V_1V_2}$ cannot be both solutions to $g_b'(x)=0$. So $g_b'(x)=0$ has at most one solution in $(0,1]$.

Note that for $i\in\{1,2,b\}$, $g_i(0)=0, g_i(1)=0, g_i(x)>0$ if $0<x<1$, and that $g_i'(x)=0$ has at most one solution in (0,1], we conclude $g_i(x)$ has one and only one maxima when $x\in(0,1)$. Let us denote this value as $\bar{p}_i$. Thus, when x increases from 0 to 1, $g_i(x)$ first increases from 0 to $\bar{p}_i$, and then decreases from $\bar{p}_i$ to 0. Therefore, when $p_i>\bar{p}_i$, Eq. (17) has no solution; when $p_i=\bar{p}_i$ or 0, Eq. (17) has only one solution; when $0<p_i<\bar{p}_i$, Eq. (17) has two solutions. In particular, we have $\bar{p}_i=g_i(\frac{\alpha_i}{\alpha_i+1})=\frac{V_i}{\alpha_i+1}\left(\frac{\alpha_i}{\alpha_i+1}\right)^{\alpha_i}$ for $i=1,2$, which completes the proof. ∎

Proof of Theorem 2: We first prove the second part of the theorem. By applying the optimization framework, we see the optimal separate sale is a solution to

$$\begin{aligned}\max_{\delta_i}\quad & U_i(\delta_i)=\sum_{i=1}^{2}\delta_i(\delta_i^{\alpha}-\delta_i^{\alpha+1}),\\ \text{s.t.}\quad & 0\le\delta_i\le 1.\end{aligned}\tag{18}$$

We can easily obtain the solution as $\delta_i^*=\frac{\alpha+1}{\alpha+2}$ and have $p_i^*=\frac{V}{\alpha+2}\left(\frac{\alpha+1}{\alpha+2}\right)^{\alpha}, U_s^*=2p_i^*\delta_i^*$. Similarly, we can solve the optimal bundling sale. By noting the definition of γ and taking the forms of U_s^* and U_b^* into Eq. (14), we can derive the profit gain ratio $\gamma(\alpha)$ as desired. By taking the derivative of $\log\gamma(\alpha)$ with respect to α, it is easy to show $\gamma(\alpha)$ is increasing in α.

Now we prove the first part of the theorem. For any given α we have $\delta_1^*=\delta_2^*=\frac{\alpha+1}{\alpha+2}$, and we denote this value as δ^*. Obviously, $\delta^*>1/2$, and we have

$$U_s^*=\delta^*\rho_1(\delta^*)H_1^{-1}(1-\delta^*)+\delta^*\rho_2(\delta^*)H_2^{-1}(1-\delta^*).\tag{19}$$

We consider a bundling sale with price p_b such that the largest equilibrium is δ^*. The service provider's utility in the optimal bundling sale satisfies

$$U_b^*\ge\delta^*\rho_b(\delta^*)H_B^{-1}(1-\delta^*).\tag{20}$$

Given the form of $H_i(x)$ and $H_B(x)$, one can easily verify that $\frac{H_i^{-1}(1-\delta^*)}{V_i}<\frac{H_B^{-1}(1-\delta^*)}{V_1+V_2}$ for $\delta^*>1/2$. Therefore, we have $H_1^{-1}(1-\delta^*)+H_2^{-1}(1-\delta^*)<H_B^{-1}(1-\delta^*)$. Since $\rho_1(\delta)=\rho_2(\delta)=\rho_b(\delta)=\delta^{\alpha}$, we have

$$\sum_{i=1}^{2}\delta^*\rho_i(\delta^*)H_i^{-1}(1-\delta^*)<\delta^*\rho_b(\delta^*)H_B^{-1}(1-\delta^*).\tag{21}$$

Combining inequalities (19), (20) and (21), we have $U_b^*>U_s^*$, therefore, the profit gain ratio $\gamma>0$. ∎

Proof of Theorem 4: Let us denote $\frac{m_1}{V_1}=\frac{m_2}{V_2}=m$. According to Lemma 3, the equilibria of separate sales satisfy $\delta_i^*=\delta_2^*$ and we denote it as δ^*. Based on the optimization formulation, we know δ^* is a solution to the following optimization:

$$\begin{aligned}\max_{\delta}\quad & U(\delta)=\delta^{\alpha+1}-\delta^{\alpha+2}-m\delta,\\ \text{s.t.}\quad & 0\le\delta\le 1.\end{aligned}\tag{22}$$

We next show $\delta^*=0$ or $\delta^*>1/2$. Note that $U(\delta)=\delta g(\delta)$ where $g(\delta)=\delta^{\alpha}-\delta^{\alpha+1}-m$. Let us consider if $\delta^*=0$ is not the unique solution, then $g(\delta^*)\ge 0$. By taking first and second order derivatives, we can derive $g(\delta)$ achieves the unique maximal value in $[0,1]$ when $\delta=\frac{\alpha}{\alpha+1}$. Let us suppose $\delta^*<\frac{\alpha}{\alpha+1}$, then $0\le g(\delta^*)<g(\frac{\alpha}{\alpha+1})$, so $\delta^*g(\delta^*)<\frac{\alpha}{\alpha+1}g(\frac{\alpha}{\alpha+1})$, which is $U(\delta^*)<U(\frac{\alpha}{\alpha+1})$. This contradicts with our assumption that $U(\delta^*)$ is the maximal value in [0,1]. Therefore, we have $\delta^*=0$ or $\delta^*\ge\frac{\alpha}{\alpha+1}>1/2$.

If $\delta^*=0$, then the optimal separate sale achieves a profit of zero, which is obviously no larger than the optimal bundling sale; if $\delta^*>1/2$, then using the same approach in the proof of Theorem 2, we can prove the optimal bundling sale is more profitable than the optimal separate sale. Combining the above two cases we prove the theorem. ∎

Proof of Theorem 5: We first analyze the bundling sale with price p_{bi} such that the equilibrium $\delta_{bi}=\delta_i^*, i=1,2$. Given the forms of $H_i(\cdot)$ and $H_B(\cdot)$, we have $\frac{H_i^{-1}(1-\delta_i^*)}{V_i}<\frac{H_B^{-1}(1-\delta_i^*)}{V_1+V_2}$. Since $p_{bi}=\delta_i^{*\alpha}H_B^{-1}(1-\delta_i^*)$ and $p_i^*=\delta_i^{*\alpha}H_1^{-1}(1-\delta_i^*)$, we have $p_{bi}>p_i^*\frac{V_1+V_2}{V_i}$, so the service provider's utility under this setting satisfies

$$U_{bi}>\left(p_i^*(V_1+V_2)/V_i-\beta_1m_1-\beta_2m_2\right)\delta_i^*.\tag{23}$$

Since the above settings $(\delta_{b1},\delta_{b2})$ are two realizations in the bundling strategy, we have the optimal bundling utility satisfies

$$\begin{aligned}2U_b^* &\ge U_{b1}+U_{b2}\\ &> p_1^*\delta_1^*(V_1+V_2)/V_1+p_2^*\delta_2^*(V_1+V_2)/V_2\\ &\quad -\beta_1(\delta_1^*+\delta_2^*)m_1-\beta_2(\delta_1^*+\delta_2^*)m_2.\end{aligned}\tag{24}$$

Therefore, we have

$$\begin{aligned}&2(U_b^*-U_s^*)>(V_2-V_1)\left(p_1^*\delta_1^*/V_1-p_2^*\delta_2^*/V_2\right)\\ &-((\beta_1-2)\delta_1^*+\beta_1\delta_2^*)m_1-((\beta_2-2)\delta_2^*+\beta_2\delta_1^*)m_2.\end{aligned}\tag{25}$$

Let us consider another service $S_0=\langle 1,0,\alpha_i\rangle$, and assume its optimal equilibrium is δ_0^*. Since the increase of unit cost reduces the value of the optimal equilibrium, and that $\delta_2^*\le\delta_1^*$, we have $\delta_0^*\ge\delta_1^*\ge\delta_2^*$. Since δ_1^*, and δ_2^* are also two sale strategies of S_0, and that δ_1^* is nearer to the optimal separate sale, we have $\frac{p_1^*\delta_1^*}{V_1}\ge\frac{p_2^*\delta_2^*}{V_2}$. Recall $V_2\ge V_1$, we have

$$(V_2-V_1)\left(p_1^*\delta_1^*/V_1-p_2^*\delta_2^*/V_2\right)\ge 0.\tag{26}$$

Since $\beta_1\le 1, \beta_2\le 1, \delta_1^*\ge\delta_2^*$, we have $((\beta_1-2)\delta_1^*+\beta_1\delta_2^*)m_1\le 0$. Since $\frac{2-\beta_2}{\beta_2}\delta_1^*\le\delta_2^*$, we have $((\beta_2-2)\delta_2^*+\beta_2\delta_1^*)m_2\le 0$. Therefore, we have

$$((\beta_1-2)\delta_1^*+\beta_1\delta_2^*)m_1+((\beta_2-2)\delta_2^*+\beta_2\delta_1^*)m_2\le 0.\tag{27}$$

Combining (25), (26) and (27), we conclude $U_b^*-U_s^*>0$. ∎